Sandpiles and river networks: Extended systems with nonlocal interactions

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A proposed continuum model of sandpile growth is able to demonstrate instantaneous long-range interactions and sudden collapses typical of real sandpiles. The model is shown to be equivalent to an evolutionary quasivariational inequality. In the limiting case of zero angle of repose the inequality describes water transport in a river network.

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INTRODUCTION

Spatially extended dissipative open systems, sandpiles being the most frequently given example, have recently attracted much interest. Under the action of external forces, these systems tend to organize themselves into a stable or statistically stable critical state which is often independent of the initial conditions and is therefore an attractor of systems dynamics.

Vast literature, starting with the works of Bak, Tang, and Wiesenfeld, has been devoted to the evolution of such systems towards the critical state and to their behavior when the criticality has been reached [1]. In cellular-automata models of dissipative transport, considered in these works, time was determined by the addition of sand grains to a pile and a new grain was added only after the relaxation, caused by the previous grains, was over. This way of introducing time common to various cellular-automata models allows them to exhibit almost instantaneous long-range interactions and sudden collapses typical of real sandpiles.

Nonlinear diffusion equations with stochastic noise [2], proposed as a continuous counterpart of cellular-automata models, are, however, unable to ensure such behavior and do not explain the evolution of the system towards a critical state [3,4].

In this work we consider a variational model of sandpile evolution, proposed in [5] and outlined in the first part of this paper. The model does demonstrate the nonlocal interactions and describes the results of experiments on building sandpiles on open supports [6]. We also discuss here a possible generalization of this model which accounts for avalanches. In the singular limiting case of zero repose angle, the model describes water transport through a river network.

I. MODEL OF SANDPILE EVOLUTION

The real process of pile growth is usually intermittent. Discharged granular material not only flows continuously over the pile slopes but is also able to build up under the charging point and then to pour suddenly down the slope in an avalanche which redistributes the material and removes the slope oversteepening.

However, the well-known almost ideally conical form of piles growing under point sources [7] suggests that these random fluctuations of pile surface occur around some mean stable evolving shape which can also be determined in a general case. The deterministic model described in this section ignores the free surface fluctuations and is proposed as a model of the mean surface evolution.

Let a cohesionless granular material having an angle of repose α be tipped out onto a rough rigid surface $y = h_0(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2)$. The form of the pile thus generated, $y = h(\mathbf{x}, t)$ is to be found.

Let us define the intensity of the source $w(\mathbf{x}, t)$ so that the volume of the randomly packed bulk material tipped out above the area $d\Omega$ in time dt be $w d\Omega dt$. The flow of granular material down the slopes of a growing pile is usually confined to a thin boundary layer which is distinctly separated from the motionless bulk [8]. Let $\vec{q}(\mathbf{x}, t)$ be the horizontal projection of the material flux in this surface layer. Assuming the bulk density of material in the pile to be constant we can write the conservation law as

$$\frac{\partial h}{\partial t} + \operatorname{div}(\vec{q}) = w.$$

We neglect the inertia and suppose that the surface flow is directed towards the steepest descent,

$$\vec{q} = -m\vec{\nabla} h,$$

where

$$m(\mathbf{x}, t) \geq 0 \tag{1}$$

is an *unknown* scalar function. The conservation law assumes now the form

$$\frac{\partial h}{\partial t} - \operatorname{div}(m\vec{\nabla} h) = w. \tag{2}$$

At $t = 0$ the free surface coincides with the support surface,

$$h|_{t=0} = h_0. \tag{3}$$

The free surface never lies below the support surface,

$$h(\mathbf{x}, t) \geq h_0(\mathbf{x}), \tag{4}$$

and wherever the free surface is above the support, it has an incline not greater then the angle of repose,

$$h(\mathbf{x}, t) > h_0(\mathbf{x}) \Rightarrow |\vec{\nabla} h(\mathbf{x}, t)| \leq \gamma, \tag{5}$$

where $\gamma = \tan(\alpha)$.

No pouring occurs over the parts of the pile surface inclined at less than the angle of repose:

$$|\vec{\nabla} h(\mathbf{x},t)| < \gamma \Rightarrow m(\mathbf{x},t) = 0. \qquad (6)$$

Let the granular material be allowed to leave the system freely through part Γ_1 of the boundary of domain Ω. The boundary condition there may be written as

$$h|_{\Gamma_1} = h_0|_{\Gamma_1} \ . \qquad (7a)$$

On the other part of the boundary, where the domain is bounded by impermeable walls, another boundary condition should be used:

$$m\frac{\partial h}{\partial n}\bigg|_{\Gamma_2} = 0. \qquad (7b)$$

The model of sandpile evolution (1)–(7) contains two unknowns, the free surface h and an auxiliary function m determining the magnitude of the free surface flux.

The conical pile on a horizontal support $h_0 = 0$ under the point source $w = w_0\delta(\mathbf{x})$ should be the first test for any sandpile evolution model. The form of such a pile in polar coordinates may be described by the function

$$h(r,t) = \gamma[R(t) - r]^+,$$

where $R(t) = (3w_0 t/\pi\gamma)^{1/3}$ is the radius of the cone base and the notation a^+ is used for $\max(a,0)$. Jointly with the function

$$m(r,t) = \frac{1}{2}\frac{dR}{dt}\left(\frac{R^2}{r} - r\right)^+$$

this forms a solution of the system (1)–(7), as long as the base of the cone is inside the domain Ω.

Direct solution of equations and inequalities (1)–(7) in a general case seems, however, difficult. Fortunately a more convenient variational formulation may be proposed.

II. QUASIVARIATIONAL INEQUALITY

The aformentioned constraint upon the incline of the free surface motivates us to seek a variational formulation for the evolutionary model of pile growth in the form of a variational or quasivariational inequality [9,10]. For every continuous function φ we define a nonlinear operator

$$B_\varphi(\psi) = \frac{1}{2}[|\vec{\nabla}\psi|^2 - M(\varphi)],$$

where

$$M(\varphi)(\mathbf{x},t) = \begin{cases} \gamma^2 & \text{if } \varphi(\mathbf{x},t) > h_0(\mathbf{x}) \\ \max[\gamma^2, |\vec{\nabla} h_0(\mathbf{x})|^2] & \text{if } \varphi(\mathbf{x},t) \le h_0(\mathbf{x}). \end{cases}$$

Let us also define a family of closed convex sets

$$K(\varphi) = \{\psi | B_\varphi(\psi) \le 0, \psi|\Gamma_1 = h_0|_{\Gamma_1}\}$$

and consider the evolutionary quasivariational inequality

find $h \in K(h)$ such that

$$\langle \partial h/\partial t - w, \varphi - h\rangle \ge 0 \text{ for every } \varphi \in K(h), \qquad (8)$$
$$h|_{t=0} = h_0,$$

where $\langle\, ,\, \rangle$ is a scalar product of two functions. The following statement holds: If there exists a function ψ_0 such that

$$\psi_0|_{\Gamma_1} = h_0|_{\Gamma_1}, \quad |\vec{\nabla}\psi_0(\mathbf{x})| < \gamma \ \text{ for all } \mathbf{x} \in \Omega \qquad (9)$$

the problem (1)–(7) is equivalent to inequality (8), that is, *function $h(\mathbf{x},t)$ is a solution of quasivariational inequality (8) if and only if there exists $m(\mathbf{x},t)$ such that the pair $\{h,m\}$ is a solution of (1)–(7).* Below we present an outline of the proof (see [5] for mathematical details).

The inequality in (8) may be formally written as an optimization problem,

$$\begin{aligned} J_h(h) = \ & \min J_h(\varphi) \\ & B_h(\varphi) \le 0 \\ & \varphi \in A \end{aligned} \qquad (10)$$

where $J_h(\varphi) = \langle \partial h/\partial t - w, \varphi\rangle$ is a linear functional and A is the set of functions satisfying (7a) on the open part of the boundary.

Let us fix the function h in J_h and B_h. Due to assumption (9), Slater's condition $(\exists\psi_0 \in A : B_h(\psi_0) < 0)$ is satisfied and the necessary and sufficient condition of optimality for (10) can be derived by the Lagrange multipliers technique [11].

Furthermore, substituting the function h into this condition, we obtain a similar condition for problem (10) (the one with the implicit constraint): the function h is a solution of quasivariational inequality (8) if and only if it satisfies the initial condition (3) and there exists a Lagrange multiplier $m(\mathbf{x},t) \ge 0$ such that the pair $\{h,m\}$ is a saddle point of Lagrangian, i.e.,

$$\begin{aligned} J_h(h) + \langle m^*, B_h(h)\rangle &\le J_h(h) + \langle m, B_h(h)\rangle \\ &\le J_h(h^*) + \langle m, B_h(h^*)\rangle \end{aligned} \qquad (11)$$

for all $h^* \in A, m^* \ge 0$. The condition of supplementary slackness

$$\langle m, B_h(h)\rangle = 0 \qquad (12)$$

is thereby fulfilled.

Let h be a solution of quasivariational inequality (8). As follows from (11), the functional

$$\left\langle \frac{\partial h}{\partial t} - w, h^*\right\rangle + \frac{1}{2}\langle m, |\vec{\nabla} h^*|^2 - M(h)\rangle$$

has a minimum on A at the point $h^* = h$. Therefore,

$$\left\langle \frac{\partial h}{\partial t} - w, \psi\right\rangle + \langle m, \vec{\nabla} h \cdot \vec{\nabla}\psi\rangle = 0 \qquad (13)$$

for every function ψ such that

$$\psi|_{\Gamma_1} = 0.$$

This is a weak formulation of Eq. (2) with boundary condition (7). Since $h \in K(h)$ condition (5) is satisfied, (6) follows from the supplementary slackness condition (12). To prove that $\{h,m\}$ is a solution of (1)–(7) we now need only to check that $h \geq h_0$.

Let us choose

$$\varphi = \begin{cases} h + (h_0 - h)^+ & \text{for } 0 \leq t \leq t_0 \\ h & \text{otherwise.} \end{cases}$$

Since $\varphi \in K(h)$ and $w \geq 0$ we obtain

$$0 \leq \left\langle \frac{\partial h}{\partial t} - w, \varphi - h \right\rangle$$
$$\leq -\frac{1}{2} \int \{[h_0(\mathbf{x}) - h(\mathbf{x}, t_0)]^+\}^2 d\Omega,$$

which proves inequality (4). (Essentially by the same argument it is possible to prove that h is a nondecreasing function of time.)

Now let $\{h, m\}$ be a solution of (1)–(7). By (5), $|\vec{\nabla} h| \leq \gamma$, where $h > h_0$. On the other part of Ω the free boundary h coincides with h_0 and therefore $h \in K(h)$. Condition (6) ensures the fulfillment of (12) which implies that the first inequality in (11) holds. To prove that h is a solution of the quasivariational inequality we have to show that the second inequality in (11) is also fulfilled.

Let $h^* \in A$. Using the weak form (13) of Eq. (2) with $\psi = h^* - h$, we obtain

$$J_h(h^*) + \langle m, B_h(h^*) \rangle - J_h(h) - \langle m, B_h(h) \rangle$$
$$= \frac{1}{2} \langle m, |\vec{\nabla}(h^* - h)|^2 \rangle \geq 0 \ ,$$

which completes the proof.

If the support surface has no steep slopes, i.e., $|\vec{\nabla} h_0| \leq \gamma$ everywhere in Ω, the problem is simplified. In this case

$$K(h) \equiv K = \{\psi \in A | \ |\vec{\nabla} \psi| \leq \gamma\}$$

and inequality (8) becomes variational. Using a modification of the penalty method [9] it is possible to prove that this inequality has a unique solution which is the limit of solutions of the nonlinear diffusion equation

$$\frac{\partial h}{\partial t} - \text{div} \left\{ \left[\frac{1}{\epsilon} (|\vec{\nabla} h|^2 - \gamma^2)^+ + \epsilon \right] \vec{\nabla} h \right\} = w, \qquad (14)$$

when the positive penalty parameter ϵ tends to zero.

In this limit, the diffusion coefficient in (14) becomes singular and it is interesting to compare our model and sandpile models with anomalous diffusion [4,12]. The high peak of the diffusion coefficient near the critical height value enables these models to demonstrate almost instantaneous long-range interactions over the regions where criticality is achieved and to simulate the evolution of subcritical states towards criticality. However, in order to obtain realistic collapselike transition to criticality from supercritical states these models should be modified — the coefficient of diffusion should tend to infinity not only in the neighborhood of the critical point but also for all supercritical height values. The limit of thus modified singular diffusion equations would be variational inequality (8) with an obstacle-type constraint,

$$K = \{\psi \in A| \ \ \psi \leq h_{\text{critical}}\}.$$

The inequality obtained is a continuous analog of the cellular-automata models with the critical height. It may be noted that, although these models are mostly studied because of their simplicity, the critical slope models are more relevant to real sandpiles [13].

Numerical solution of quasivariational inequality (8) was considered in [5,14]. Here we obtain analytical solutions describing the growth of real sandpiles on open platforms.

III. SANDPILES ON OPEN SUPPORTS

Experiments on the buildup of sandpiles on flat horizontal open platforms of different shapes have been recently described by Puhl [6]. This work shows that the buildup of a pile on an empty support due to addition of grains from a point source yields a cone which grows until its base touches the boundary. When this happens, a runway appears and almost all added grains move down this way to the edge. At this point the growth of the pile practically stops; the final form of the pile does not depend on the shape of the support.

On the other hand, the form of a pile obtained by first putting a huge amount of sand on a support and letting it then evolve depends strongly on the shape of the support. For circular supports, Puhl obtained perfect cones and for square and octagonal supports, pyramids with a respective base.

Let us show that this is exactly the behavior predicted by our model. Since the variational inequality (8) is an equivalent formulation of the model (1)–(7), the unique solution for a point source $w_0 \delta(\mathbf{x} - \mathbf{x}_0)$ is a growing cone provided its base is inside Ω. Let the cone touch the boundary at time t_0. At this moment a generator inclined at the angle of repose connects the cone apex with a point on the open boundary. A further increase of $h(\mathbf{x}_0)$ is obviously impossible and, since h is a nondecreasing function of time, $h(\mathbf{x}_0, t) = \text{const}$ for $t \geq t_0$. Function $\varphi_0(\mathbf{x}, t) = h(\mathbf{x}, \min(t, t_0))$ belongs to the set K; from variational inequality (8) we obtain

$$0 \leq \left\langle \frac{\partial h}{\partial t} - w_0 \delta(\mathbf{x} - \mathbf{x}_0), \varphi_0 - h \right\rangle$$
$$= -\frac{1}{2} \int_{t_0}^{T} \frac{d}{dt} \|h(\mathbf{x}, t) - h(\mathbf{x}, t_0)\|^2 dt$$
$$-w_0 \int_{t_0}^{T} (h(\mathbf{x}_0, t_0) - h(\mathbf{x}_0, t)) dt$$
$$= -\frac{1}{2} \|h(\mathbf{x}, T) - h(\mathbf{x}, t_0)\|^2,$$

which proves that at $t = t_0$ the growth of the pile stops.

To simulate Puhl's experiments of the second type one may consider the case of a stationary distributed source with intensity $w(\mathbf{x})$ everywhere positive. Since the growth of a pile on an open platform is bounded and monotonous there exists a stationary solution. In this steady state, all sand discharged above any point of Ω will pour down the surface to the boundary and this is possible only if $|\vec{\nabla}h| = \gamma$ almost everywhere. This equation possesses an infinite set of solutions which are zero on the boundary $\partial\Omega$. However, the variational inequality provides the additional condition

$$\langle -w, \varphi - h\rangle \geq 0 \text{ for every } \varphi \in K,$$

and thus the stationary solution sought is the one that maximizes $\int hwd\Omega$. Since $w > 0$, this solution also maximizes $\int hd\Omega$, which brings us to the well-known problem of the completely plastic torsion of a beam [15]. The unique solution of this problem

$$h(\mathbf{x}) = \gamma \mathrm{dist}(\mathbf{x}, \partial\Omega)$$

describes the pile forms in Puhl's experiment.

IV. AVALANCHES

The formation of stockpiles of sorted crushed stone was observed at a quarry near Beer Sheva. Though stockpiles were periodically affected by the reclaiming of the bulk material it was still possible to receive a qualitative picture of avalanches during the building of a cone by the fall of stone debris from a conveyor belt at a low rate.

In the vicinity of the pile apex there was a region of continuous flow. Rock fragments discharged from a small height usually rolled a short distance down the slope and were trapped into the bulk. The accumulation of material occurred mainly on the upper part of the cone. At times the upper part would settle gently, advancing the bulk material further down the slope. Most of these advancements were, however, small and stopped quickly; these events may possibly be attributed to macroscopic discontinuous densifications described in [16].

Mass avalanches more often started as a slide of a large section of the pile slope. It was usually difficult to single out the direct cause or location of the flow initiation — a part of the surface was coming into the motion simultaneously like a solid block. Fast destruction of the block during the sliding usually proceeded from the block boundaries; sometimes a seemingly rigid island was seen to be sliding in the middle of a chaotical flow of particles. These large avalanches were able to travel long distances incorporating new particles on their way into the motion. Often these avalanches reached the foot of a cone and it was then possible to observe the process of formation of a new surface layer, as the boundary between the supported and thus immobilized particles and those still moving propagated quickly up the slope from the foot of the pile.

To incorporate such avalanches into the model of pile growth one must take into account the fact that a real granular material should be characterized by two angles — the minimal angle of repose α_r and the maximal angle of stability α_m [8]. As long as the incline of the pile surface is less than α_r there is no surface flow; slopes steeper than α_m are unstable. Between α_r and α_m there is a region of bistable behavior where the material is either stationary or flowing.

Let us now assume the limiting slope angle in the model (8) to be a function of $\mathbf{x}$ and t,

$$\alpha(\mathbf{x}, t) \in [\alpha_r, \alpha_m].$$

If α were not dependent on time, the model would still describe uninterrupted monotonic growth of a pile. However, a sudden local decrease of the limiting angle in a small region may cause an instantaneous collapse-like unlocal change of the solution. (The reader may consider a simple conical pile to see why a local instantaneous change of the limiting angle should yield a nonlocal rearrangement of the entire pile surface.) This model may be justified by the following physical arguments.

Stability of a randomly packed pile is ensured by a stress carrying continuous net of particles [8]. When the load exceeds a local threshold value a rearrangement of particles takes place. Near the free surface there are no strong obstacles to the dilation needed for a mass rearrangement. The dilation immediately decreases the angle of shearing resistance which depends on the packing density and determines the maximal allowed slope incline [16]. The upper part of the slope may become unsupported and slide downwards.

Particles rolling or sliding down the slope transfer their momentum to the motionless particles of the surface layer, cause vibration and make this layer less stable. The dependence of the limiting slope incline on the surface flux is confirmed by experiment [17] — after a big avalanche, the slope is always inclined at the minimal angle α_r, which is not necessarily so after small avalanches. The magnitude of this flux can therefore affect the value of $\alpha(\mathbf{x}, t)$.

Actual simulation of sandpile avalanches should, in our opinion, require a better understanding of the underlying processes. We believe, however, that our continuous model is able, in principle, to describe collapses and nonlocal interactions observed in sandpiles and similar systems. It may be noted that cellular-automata models with a variable and flux-dependent limiting angle have also been recently proposed [18].

V. RIVER NETWORKS

River networks, one of the most common of nature's fractal patterns, are extended dissipative systems which have also been extensively studied. Lattice models for the evolution of these networks [19] yield pictures of river nets resembling hydrological maps. Similar lattice models have been used for automatic derivation of hydrological maps from digital elevation data [20]. As is shown below, a continuum model of water transport through a river network may be obtained as a special case of the sandpile model.

Let h_0 be the land surface and w the intensity of precipitation. We assume for simplicity that the water neither evaporates nor penetrates the soil but just flows down the slopes and accumulates into lakes at local depressions of the land surface. The level of a lake rises until the water reaches a divide of two basins. Then a river running out of the lake is generated and all additional water that comes into the lake is transferred to another lake below. Under open boundary conditions, such a system of lakes and rivers will organize itself into a critical steady state where all the precipitating water is removed through the boundary. During this evolution, the effect of the creation of a new river or of a local change of precipitation intensity is transferred almost instantaneously across the parts of the system that are already in a critical state. The local water flux or a lake's growth rate are therefore determined not only by the local conditions but also by some distant events that may occur at the same time.

Since the free surface in this problem either coincides with the land surface or, where it is higher, is the horizontal surface of the lake, it may be expected that this surface could be described by the quasivariational inequality (8) with a zero angle of repose. This case, however, needs a special consideration since the flow in the lakes is not confined to a thin boundary layer and its direction is not determined by the free surface gradient which is there zero.

Let $\vec{v} = (v_{x_1}, v_{x_2}, v_y)$ be the water flow velocity (in a lake or in a very thin surface layer over the slope). Integrating the continuity equation

$$\frac{\partial v_{x_1}}{\partial x_1} + \frac{\partial v_{x_2}}{\partial x_2} + \frac{\partial v_y}{\partial y} = 0$$

with respect to y yields

$$\begin{aligned} 0 &= \int_{h_0}^{h} \frac{\partial v_{x_1}}{\partial x_1} dy + \int_{h_0}^{h} \frac{\partial v_{x_2}}{\partial x_2} dy + v_y|_{h_0}^{h} \\ &= \frac{\partial}{\partial x_1} \int_{h_0}^{h} v_{x_1} dy + \frac{\partial}{\partial x_2} \int_{h_0}^{h} v_{x_2} dy \\ &\quad - v_n|_h \sqrt{1 + |\vec{\nabla} h|^2} + v_n|_{h_0} \sqrt{1 + |\vec{\nabla} h_0|^2}\ , \end{aligned}$$

where normal velocities v_n are determined from the kinematic boundary conditions

$$v_n|_{h_0} = 0, \quad v_n|_h - v_n^s = w^s.$$

Here $v_n^s = -(\partial h/\partial t)/\sqrt{1 + |\vec{\nabla} h|^2}$ is the free surface velocity and $w^s = w/\sqrt{1 + |\vec{\nabla} h|^2}$ is the intensity of water influx through the free surface. Combining these equations we arrive at the same conservation law as before,

$$\frac{\partial h}{\partial t} + \operatorname{div}(\vec{q}) = w,$$

where $\vec{q}$ is the horizontal projection of the water flux. The direction of this vector is now determined by the relations $\vec{q} = -m\vec{\nabla} h$ and $m(\mathbf{x}, t) \geq 0$ only over the slopes since, in the lakes, where $\vec{\nabla} h = \vec{0}$, the hydrodynamics are obviously nontrivial. The flow in the lake does not affect the free surface and we do not need to specify its direction. This is a modification of the previous model implied by the physics of the problem. Note that Slater's condition, which we used in our proof of the equivalence of differential and variational formulations, is no longer true for the zero angle of repose. Nevertheless, as we show below, in this case the weaker modified formulation also yields the quasivariational inequality (8).

Let $\gamma = 0$ and $\{h, m\}$ be a solution of the modified model (1)–(5) and (7). Then $h \in K(h)$ and for any function $\varphi \in K(h)$,

$$\begin{aligned} \left\langle \frac{\partial h}{\partial t} - w, \varphi - h \right\rangle &= -\langle \operatorname{div}\vec{q}, \varphi - h\rangle \\ &= -\int_0^T \int_{\partial\Omega} q_n(\varphi - h) \\ &\quad + \int_0^T \int_{\Omega} \vec{q} \cdot \vec{\nabla}(\varphi - h). \end{aligned}$$

The first integral on the right-hand side is zero due to the boundary conditions. Gradients $\vec{\nabla} h$ and $\vec{\nabla}\varphi$ are both zero on the part of Ω where $h > h_0$ and on the other part of this domain $\vec{q} = -m\vec{\nabla} h_0$, $m \geq 0$, and $|\vec{\nabla}\varphi| \leq |\vec{\nabla} h_0|$, therefore

$$\vec{q} \cdot \vec{\nabla}(\varphi - h) = -m(\vec{\nabla} h_0 \cdot \vec{\nabla}\varphi - |\vec{\nabla} h_0|^2) \geq 0,$$

and thus the quasivariational inequality holds.

The free boundary h can be found from this inequality. However, this is only a part of the solution — it is the flux $\vec{q}$, the dual variable, which is of interest in geomorphological and hydrological applications. The determination of the free surface fluxes is also necessary in problems of bulk solids mechanics of polydisperse materials [14].

At time t, the water flux down the slopes is defined in the coincidence set

$$\Omega_0 = \{\mathbf{x} \in \Omega |\ h(\mathbf{x}, t) = h_0(\mathbf{x})\}.$$

In this domain $\partial h/\partial t = 0$ and $m = q/|\vec{\nabla} h_0|$, where $q = |\vec{q}|$ and we suppose that $\vec{\nabla} h_0 \neq \vec{0}$ almost everywhere. The equation of water balance (2) in this region yields

$$-\operatorname{div}\left(q \frac{\vec{\nabla} h_0}{|\vec{\nabla} h_0|}\right) = w, \qquad \mathbf{x} \in \Omega_0. \tag{15}$$

Provided the free boundary h has been found from the quasivariational inequality and thus Ω_0 is known, the hyperbolic equation (15) may be used for the flux calculation. Since the characteristics of this equation are the lines of steepest descent of the land surface, the boundary condition for (15) should specify the flux on those parts of $\partial\Omega_0$ where vector $-\vec{\nabla} h_0$ is directed inside domain Ω_0.

Equation (15) has been known for a long time [21]. Nevertheless, various cellular-automata models or topological methods based on the evaluation of drainage areas have usually been used instead of it for the determination of water fluxes [19,20]. Below we present an illustrating example in which we determine the fluxes by solving Eq. (15) using the finite-element approximation for a simple topography without lakes (in this case $\Omega_0 = \Omega$).

Domain Ω has been triangulated and the land surface is assumed to be linear inside each finite element. Water gain by precipitation has been related to the centers of the elements; the flux values have been determined in the centers and vertices of elements. The outflow of an element has been directed towards its lowest node or towards the neighboring element across their common edge, depending on the local topography. The outflow of a node has also been directed either towards a lower node along the common edge or into one of the neighbor-

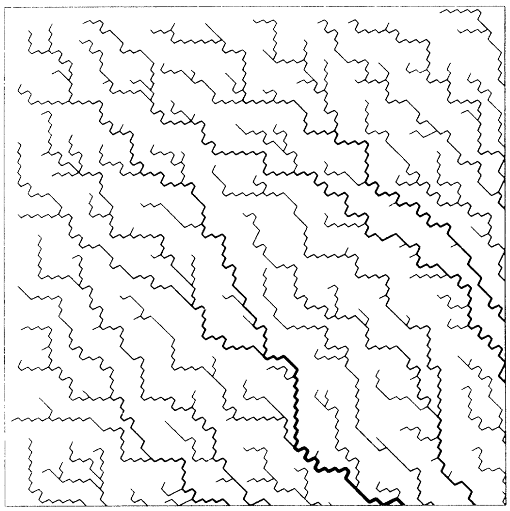

FIG. 1. River network on a randomly perturbed inclined flat surface. The regular finite element net 50 × 50 is used; rivers with drainage areas not less than ten finite elements are shown. The river's width in the figure is proportional to the drainage area.

ing elements. All flows directed into an element join at its center. The resulting system of linear balance equations is sparse and can be effectively solved even for a large number of elements.

To illustrate the numerical solution we used a flat surface inclined towards the southeast and added a small random noise to the heights in the nodes of a regular finite element net. Precipitation intensity was taken to be uniform so that the water fluxes were proportional to the drainage areas. The calculated rivers with drainage areas larger than a certain fixed value are shown in Fig. 1. Generalization of this numerical procedure, allowing for the presence of lakes, is also possible.

CONCLUSION

A model describing the evolution of sandpiles and water transport in river nets has been proposed. In this model, no constitutive equation relating the rate of surface transport to the excess of the slope incline is used and it is not even assumed that the surface flux is a local functional of the free surface. The local topography of this surface determines only the direction of surface transport but not the flux value which may be influenced by distant events.

The rate of surface transport in our model is determined by a Lagrange multiplier, which ensures the fulfilment of a mass conservation law in the presence of a constraint and depends on the free surface and the external source in a nonlocal way. That is why the model is able to account for long-range instantaneous interactions over those regions where criticality has been reached, which is a crucial feature typical of extended dissipative systems.

In transition to the variational formulation of the model in the form of a quasivariational inequality, this dual variable disappears and the free boundary may be found from this inequality without the determination of the surface flux.

The knowledge of the surface flux is, however, needed in various applications. The conservation law can be used for its calculation provided that the free surface has been already found.

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